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Bayesian influence analysis: a geometric approach

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Summary

In this paper we develop a general framework of Bayesian influence analysis for assessing various perturbation schemes to the data, the prior and the sampling distribution for a class of statistical models. We introduce a perturbation model to characterize these various perturbation schemes. We develop a geometric framework, called the Bayesian perturbation manifold, and use its associated geometric quantities including the metric tensor and geodesic to characterize the intrinsic structure of the perturbation model. We develop intrinsic influence measures and local influence measures based on the Bayesian perturbation manifold to quantify the effect of various perturbations to statistical models. Theoretical and numerical examples are examined to highlight the broad spectrum of applications of this local influence method in a formal Bayesian analysis.

Some key words: Influence measure; Perturbation manifold; Perturbation model; Prior distribution.

1. Introduction

A formal Bayesian analysis of data $z = (z_1, \ldots, z_n)$ involves the specification of a sampling distribution $p(z \mid \theta)$ and a prior distribution $p(\theta)$, where $\theta = (\theta_1, \ldots, \theta_k)^{\mathrm{T}}$ represents the parameters of inferential interest and varies in an open set Θ of R^k. To carry out Bayesian inference, we usually use Markov chain Monte Carlo methods to simulate samples from the posterior distribution $p(\theta \mid z)$, which is proportional to $p(z \mid \theta)p(\theta)$. Subsequently, we can calculate posterior quantities of θ in R^k, such as the posterior mean $M(h) = \int h(\theta)p(\theta \mid z)d\theta$ of a function $h(\theta)$. For notational simplicity, we do not emphasize the dominating measure explicitly throughout the paper. There is a great deal of interest in the degree to which posterior inferences are sensitive to $p(\theta)$, $p(z \mid \theta)$ and $(z_1, \ldots, z_n)$ (Kass et al., 1989; McCulloch, 1989; Berger, 1990, 1994; Dey et al., 1996; Gustafson, 2000; Sivaganesan, 2000; Oakley & O'Hagan, 2004).

There are three major formal influence techniques, including case influence measures and global and local robustness approaches, for quantifying the degree of dependence of the posterior distribution on these three key elements of Bayesian analysis including the prior, the sampling distribution and the data (Berger, 1990, 1994). In Bayesian analysis, case influence measures primarily calculate the influence of a set of observations in order to identify outliers and influential observations. Most case influence measures are based on the posterior and/or predictive distribution through either case deletion or perturbation (Guttman & Peña, 1993; Peña & Guttman, 1993; Carlin & Polson, 1991; Peng & Dey, 1995). For instance, several case influence diagnostics have been developed to quantify the possible outlyingness of a set of observations based on mean-shift or variance-shift models (Guttman & Peña, 1993; Peña & Guttman, 1993).

The key idea of the global robustness approach is to compute a range of posterior quantities as the perturbation to each of the three key elements changes in a certain set of distributions, and then determine the extremal ones. There are drawbacks with this approach, including the scale chosen for the posterior quantities, the size of the perturbation and its limitation to linear functionals as well as simplicity of models. To address the scale issue, several scaled versions of the range have been proposed for the prior perturbation class (Ruggeri & Sivaganesan, 2000).

The local robustness approach primarily computes the derivatives of posterior quantities with respect to a minor perturbation to $p(\theta)$ and $p(z \mid \theta)$. In the frequentist literature, Cook's (1986) influence approach is particularly useful for perturbing $p(z \mid \theta)$ in order to detect influential observations and assess model misspecification in parametric and semiparametric models (Zhu & Lee, 2001; Zhu et al., 2007). McCulloch (1989) further extends the local influence approach of Cook (1986) to assess the effects of perturbing the prior in a Bayesian analysis. In the Bayesian literature, several analogues of local influence have been developed using either the curvature of influence measures (Lavine, 1992; Dey & Birmiwal, 1994; Millar & Stewart, 2007; Van der Linde, 2007) or the Fréchet derivative of the posterior with respect to the prior (Berger, 1994; Gustafson & Wasserman, 1995; Dey et al., 1996; Gustafson, 1996; Berger et al., 2000). Very little has been done on developing general Bayesian influence analysis methods for simultaneously perturbing z, $p(\theta)$ and $p(z \mid \theta)$, assessing their effects and examining their applications in statistical models (Berger et al., 2000). To our knowledge, Clarke & Gustafson (1998) is one of the few papers on simultaneously perturbing $\{z, p(\theta), p(z \mid \theta)\}$ in the context of independent and identically distributed data.

A key motivation for the proposed methodology is to unify influence concepts for many complex Bayesian models, for which very few or no methods exist, so that the effects of different perturbations can be identified. These models include many Bayesian parametric and semiparametric models, perhaps with missing data; see the Supplementary Material. Our development includes formal assessment of outliers and influential points as well as sensitivity analyses regarding the three major components of the Bayesian model: the prior, sampling distribution, and the data. For instance, sensitivity to the data can be evaluated by perturbing all the data points by random noise, redoing the analysis, and getting a spectrum of different inferences defined by noise (Wang et al., 2009; Clarke, 2010).

2. THE BAYESIAN PERTURBATION MODEL AND MANIFOLD

2·1. *The Bayesian perturbation model*

We develop a Bayesian model to characterize various perturbation schemes to z, $p(z \mid \theta)$ and $p(\theta)$. We introduce perturbations into the model $p(z, \theta) = p(z \mid \theta)p(\theta)$ through a vector $\omega = \omega(z, \theta)$, which varies in a set Ω. That is, ω is a mapping from the product space of the sample space $\mathcal{Z}$ and the parameter space Θ to Ω. Generally, ω includes many perturbation schemes including the additive ϵ-contamination class to the prior as detailed below. Moreover, ω must be chosen carefully so that the perturbation is meaningful and sensible.

Let $p(z, \theta \mid \omega)$ be the probability density of (z, θ) for the perturbed model. We assume that the probability measures of $p(z, \theta \mid \omega)$ for all $\omega \in \Omega$ have a common dominating measure and that there is an $\omega^0 \in \Omega$ such that $p(z, \theta \mid \omega^0) = p(z, \theta)$ for all (z, θ). We refer to $p(z, \theta \mid \omega^0) = p(z, \theta)$

as the baseline joint distribution, where ω^0 can be regarded as the central point of Ω representing no perturbation. We define the Bayesian perturbation model $\mathcal{M}$ as a family of probability densities $p(z, \theta \mid \omega)$ as ω varies in Ω. The Bayesian perturbation model includes individual perturbation schemes to z, $p(\theta)$ and $p(z \mid \theta)$, and their combinations. We focus on each individual scheme as follows.

Example 1. The Bayesian perturbation model for the prior includes many existing schemes, such as the additive ϵ-contamination class and the linear and nonlinear perturbation classes. For instance, the additive ϵ-contamination scheme is given by $p(\theta \mid \omega) = p(\theta) + \lambda\{g(\theta) - p(\theta)\}$, where $\lambda \in [0, 1]$ and $g(\theta)$ belongs to a class of contaminating distributions, denoted by $\mathcal{G}$ (Berger, 1994; Dey & Birmiwal, 1994). In this case, $\Omega = \{\omega = \lambda\{g(\theta) - p(\theta)\} : (\lambda, g(\cdot)) \in [0, 1] \times \mathcal{G}\}$ and $\omega(z, \theta)$ are independent of the data. Thus, $\omega^0 = 0$ and $p(z, \theta \mid \omega) = p(z \mid \theta)p(\theta \mid \omega)$.

Example 2. The Bayesian perturbation model for the data includes many perturbation schemes to individual data observations of z (Cook, 1986; Guttman & Peña, 1993; Peña & Guttman, 1993; Zhu et al., 2007). The perturbation scheme to data points is proposed for identifying outliers and influential observations. As an illustration, we consider the standard linear regression model $y_i = x_i^{\mathrm{T}}\beta + \epsilon_i$, where x_i is a $p \times 1$ covariate vector, β is a $p \times 1$ vector of regression coefficients and the ϵ_i are independently and identically distributed $N(0, \sigma^2)$ random variables. Let c_l be an $l \times 1$ vector with all elements equal to c for a fixed scalar c and an integer l, written as 1_n, 1_p and 0_m. A perturbation scheme to perturb the covariate x_i is given by $x_i(\omega_i) = x_i + \omega_i 1_p$. In this case, $z_i = (y_i, x_i^{\mathrm{T}})^{\mathrm{T}}$, $\theta = (\beta^{\mathrm{T}}, \sigma^2)^{\mathrm{T}}$, $\omega = (\omega_1, \ldots, \omega_n)^{\mathrm{T}}$, $\omega^0 = 0_n$ and Ω is a subset of R^n. An alternative perturbation scheme to the linear regression model is the well-known mean shift model (Guttman & Peña, 1993; Peña & Guttman, 1993). It is assumed that $y_i = x_i^{\mathrm{T}}\beta + \omega_i + \epsilon_i$ for i in a set of k distinct integers chosen from the set $\{1, \ldots, n\}$, denoted by $I = \{i_1, \ldots, i_k\}$, and $y_i = x_i^{\mathrm{T}}\beta + \epsilon_i$ for all other is. In this case, the perturbation scheme is $\omega = (\omega_{i_1}, \ldots, \omega_{i_k})^{\mathrm{T}}$ and $\omega^0 = 0_k$. Another important scheme is a geometric mixture model for case deletion or case weights (Millar & Stewart, 2007; Van der Linde, 2007). Specifically, let $q(z_i)$ be an arbitrary density of z_i independent of θ, then the geometric mixture model for perturbing the ith observation is given by $p(z \mid \theta, \omega) = \{\prod_{j \neq i} p(z_j \mid \theta)\}p(z_i \mid \theta)^{\lambda}q(z_i)^{1-\lambda}/\{\int p(z_i \mid \theta)^{\lambda}q(z_i)^{1-\lambda}dz_i\}$, where $\omega = \lambda$ varies in $[0, 1]$ and $p(z_i \mid \theta)$ is the density of z_i under the linear model assumption. In this case, $\omega^0 = 1$ represents no perturbation. When $\lambda = 0$, $p(z_i \mid \theta)$ disappears in $p(z \mid \theta, 0)$, which is equivalent to deleting z_i.

Example 3. The Bayesian perturbation model for the sampling distribution includes many perturbation schemes to $p(z \mid \theta)$ such as the additive ϵ-contamination class. We may also consider a class of perturbed sampling distributions $p(z \mid \theta, \omega)$ defined by

$$p(z \mid \theta, \omega) = p(z \mid \theta) \exp\left\{\sum_{j=1}^{m} \omega_j u_j(z; \theta) - 0{\cdot}5 \sum_{j=1}^{m} \omega_j^2 u_j(z; \theta)^2 - C(\theta, \omega)\right\}, \qquad (1)$$

where $C(\theta, \omega)$ is the normalizing constant, $\omega = (\omega_1, \ldots, \omega_m)^{\mathrm{T}}$ is an $m \times 1$ vector and $u_j(z; \theta)$ is a fixed scalar function having zero mean under $p(z \mid \theta)$. In this case, $\omega^0 = 0_m$ represents no perturbation. The number m in the perturbation (1) can either be as small as 1 or can increase with n (Copas & Eguchi, 2005; Zhu et al., 2007).

2·2. *The Bayesian perturbation manifold*

We develop a new geometric framework, called a Bayesian perturbation manifold, to measure each perturbation ω in the Bayesian perturbation model. Based on this manifold, we are able to measure the amount of perturbation, the extent to which each component of a perturbation model contributes to $p(z, \theta)$ and the degree of orthogonality for the components of the perturbation model. Such a quantification is useful for rigorously assessing the relative influence of each component in the Bayesian analysis, and can reveal any discrepancies among the data, the prior or the sampling model.

For an infinite dimensional set Ω, we assume throughout the paper that $\mathcal{M}$ forms a Riemannian Hilbert manifold (Friedrich, 1991; Lang, 1995) under some regularity conditions. For a given $p(z, \theta \mid \omega) \in \mathcal{M}$, we consider a smooth curve $C(t) = p\{z, \theta \mid \omega(t)\}$ through the space of perturbation models $\mathcal{M}$ with open interval domains containing 0 and $p\{z, \theta \mid \omega(0)\} = p(z, \theta \mid \omega)$. Note that ω may be different from ω^0. We require $C(t)$ to be smooth enough such that $\dot{\ell}\{z, \theta \mid \omega(t)\} = d \log p\{z, \theta \mid \omega(t)\}/dt$, called the tangent or derivative vector, exists with $\int \dot{\ell}\{z, \theta \mid \omega(t)\}^2 p\{z, \theta \mid \omega(t)\} dz d\theta < \infty$ for all t in the open interval domain. Since $p\{z, \theta \mid \omega(t)\}$ is the joint density of (z, θ) given $\omega(t)$, that is $\int p\{z, \theta \mid \omega(t)\} dz d\theta = 1$, the tangent space of $\mathcal{M}$ at ω, denoted by $T_\omega \mathcal{M}$, is formed by the tangent vectors $\dot{\ell}\{z, \theta \mid \omega(0)\}$ for all possible smooth curves $C(t)$ such that $\int \dot{\ell}\{z, \theta \mid \omega(0)\} p\{z, \theta \mid \omega(0)\} dz d\theta = 0$. We can introduce the inner product of any two tangent vectors $v_1(\omega)$ and $v_2(\omega)$ in $T_\omega \mathcal{M}$ as

$$< v_1, v_2 > (\omega) = \int \{v_1(\omega) v_2(\omega)\} p(z, \theta \mid \omega) dz d\theta. \tag{2}$$

When ω varies in a Euclidean space and is independent of z and θ, the inner product $< v_1, v_2 > (\omega)$ in (2) is closely associated with the Fisher information. See Example 6 for details. Thus, the squared length $||v(\omega)||^2$ of a tangent vector $v(\omega) \in T_\omega \mathcal{M}$ is $< v, v > (\omega) = \int v(\omega)^2 p(z, \theta \mid \omega) dz d\theta$. The length of the curve $C(t)$ from t_1 to t_2 is

$$S_C\{\omega(t_1), \omega(t_2)\} = \int_{t_1}^{t_2} [< \dot{\ell}\{z, \theta \mid \omega(t)\}, \dot{\ell}\{z, \theta \mid \omega(t)\} > \{\omega(t)\}]^{1/2} dt. \tag{3}$$

Next, we need to introduce the concept of a geodesic, which is a direct extension of the straight line in Euclidean space, on $\mathcal{M}$. Consider a real function $f(\omega)$ defined on $\mathcal{M}$ and a smooth curve $p\{z, \theta \mid \omega(t)\}$ in $\mathcal{M}$ with $p\{z, \theta \mid \omega(0)\} = p(z, \theta \mid \omega)$ and $\dot{\ell}\{z, \theta \mid \omega(0)\} = v(\omega)$. We define $df[v](\omega) = \lim_{t \to 0} t^{-1}(f[p\{z, \theta \mid \omega(t)\}] - f[p\{z, \theta \mid \omega(0)\}])$ as the directional derivative of f at the perturbation distribution $p(z, \theta \mid \omega)$ in the direction of $v(\omega) \in T_\omega \mathcal{M}$. We consider two smooth vector fields $u(\omega)$ and $v(\omega)$, which are not only the tangent vectors in $T_\omega \mathcal{M}$, but also smooth functions of ω in Ω. We define the directional derivative of a vector field $u(\omega)$ in the direction of $v(\omega)$, called the connection, which is given by $du[v](\omega) = \lim_{t \to 0} t^{-1}[u\{\omega(t)\} - u\{\omega(0)\}]$. Intuitively, if ω varies in a Euclidean space, then $du[v](\omega)$ is closely associated with the second derivative of $\ell(z, \theta \mid \omega)$ with respect to ω. We consider the Levi–Civita connection, which has several nice geometric properties (Amari, 1990; Lang, 1995) and is given by

$$\nabla_v u(\omega) = du[v](\omega) - 0{\cdot}5\{u(\omega) v(\omega) p(z, \theta \mid \omega) - \int u(\omega) v(\omega) p(z, \theta \mid \omega) dz d\theta\}.$$

A geodesic with respect to the Levi–Civita connection on $\mathcal{M}$ is a smooth curve $\gamma(t) = p\{z, \theta \mid \omega(t)\}$ on $\mathcal{M}$ with open interval domain (a, b) and $\dot{\ell}\{z, \theta \mid \omega(t)\} = v\{\omega(t)\}$ such that the Levi–Civita connection $\nabla_v v\{\omega(t)\} = 0$. Intuitively speaking, as one moves tangent vectors of a geodesic along the same geodesic, one can keep them pointing in the same direction. Moreover, geodesics can be interpreted as the shortest local path between points on $\mathcal{M}$. For a fixed perturbation distribution $p(z, \theta \mid \omega)$ and a given direction of $v(\omega) \in T_\omega\mathcal{M}$, there is a unique geodesic $\gamma(t) = p\{z, \theta \mid \omega(t)\}$ with open interval domains covering 0 such that $\gamma(0) = p(z, \theta \mid \omega)$ and $\dot{\gamma}(0) = v(\omega)$. Finally, based on these geometric quantities of $\mathcal{M}$, we introduce the definition of a Bayesian perturbation manifold.

Definition 1. *A Bayesian perturbation manifold* $(\mathcal{M}, < u, v >, \nabla_v u)$ *is the manifold* $\mathcal{M}$ *with an inner product* $< u, v >$ *and the Levi–Civita connection* $\nabla_v u$.

When Ω is an open set of R^m, under some regularity conditions, the Bayesian perturbation manifold is an m-dimensional manifold (Amari, 1990, p. 16; Kass & Vos, 1997; Zhu et al., 2007). Now, we examine some examples of Bayesian perturbation manifolds based on several perturbations to the data, the prior, and the sampling distribution.

Example 1, *continued*. We consider the Bayesian perturbation model for the ϵ-contamination class to the prior given by $\mathcal{M} = \{\{(1-\lambda)p(\theta) + \lambda g(\theta)\}p(z \mid \theta) : \lambda \in [0, 1], g(\cdot) \in \mathcal{G}\}$. In this case, $\omega(t) = t\{g(\theta) - p(\theta)\}$ for a given $g(\cdot) \in \mathcal{G}$, and therefore we consider the smooth curve $C_g(t) = p\{z, \theta \mid \omega(t)\} = [p(\theta) + t\{g(\theta) - p(\theta)\}]p(z \mid \theta)$. It can be shown that $v_g\{\omega(t)\} = \dot{\ell}\{z, \theta \mid \omega(t)\} = \{g(\theta) - p(\theta)\}/[p(\theta) + t\{g(\theta) - p(\theta)\}]$. For any two densities $g_1(\cdot)$ and $g_2(\cdot)$ in $\mathcal{G}$, we can calculate the tangent vectors $v_{g_i}\{\omega(0)\} = \{g_i(\theta) - p(\theta)\}\{p(\theta)\}^{-1}$ for $i = 1, 2$ and their inner product as

$$< v_{g_1}, v_{g_2} > (\omega^0) = \int [g_1(\theta)\{p(\theta)\}^{-1} - 1][g_2(\theta)\{p(\theta)\}^{-1} - 1]p(\theta)d\theta,$$

which is also independent of $p(z \mid \theta)$. In particular, $< v_g, v_g > (\omega^0) = \int \{g(\theta)/p(\theta) - 1\}^2 p(\theta)d\theta$ reduces to the L^2 norm considered in Gustafson (1996).

We further consider a Bayesian perturbation model for the sole perturbation scheme to hyperparameters of the prior given by $\mathcal{M} = \{p(z, \theta \mid \omega) = p(\theta \mid \omega)p(z \mid \theta) : \omega = (\omega_1, \ldots, \omega_m)^{\mathrm{T}}\}$, in which ω is independent of both z and θ. Let $\omega(t) = (\omega_1, \ldots, \omega_{j-1}, \omega_j + t, \omega_{j+1}, \ldots, \omega_m)^{\mathrm{T}}$, $\ell(\theta \mid \omega) = \log p(\theta \mid \omega)$ and $\omega_k(t)$ be the kth component of $\omega(t)$. Since $\ell(z, \theta \mid \omega) = \log p(\theta \mid \omega) + \log p(z \mid \theta)$, we have

$$\dot{\ell}\{z, \theta \mid \omega(0)\} = d\ell\{z, \theta \mid \omega(t)\}/dt \mid_{t=0} = \sum_{k=1}^{m} [\dot{\omega}_k(t)\partial_{\omega_k}\ell\{\theta \mid \omega(t)\}] \mid_{t=0} = \partial_{\omega_j}\ell(\theta \mid \omega),$$

where $\dot{\omega}_k(t) = d\omega_k(t)/dt$ and $\partial_{\omega_j} = \partial/\partial\omega_j$. Therefore, $T_\omega\mathcal{M}$ is spanned by the m functions $\partial_{\omega_j}\ell(\theta \mid \omega)$ pointwise in ω. Since $\int p(z \mid \theta)dz = 1$, the inner product between $\partial_{\omega_j}\ell(\theta \mid \omega)$ and $\partial_{\omega_k}\ell(\theta \mid \omega)$, denoted by $G_{jk}(\omega)$, is given by

$$\begin{aligned} G_{jk}(\omega) &= \int \partial_{\omega_j}\ell(\theta \mid \omega)\partial_{\omega_k}\ell(\theta \mid \omega)p(\theta \mid \omega)p(z \mid \theta)dzd\theta \\ &= \int \partial_{\omega_j}\ell(\theta \mid \omega)\partial_{\omega_k}\ell(\theta \mid \omega)p(\theta \mid \omega)d\theta, \end{aligned} \tag{4}$$

which is independent of $p(z \mid \theta)$.

Furthermore, suppose that $p(\theta) = p(\theta_1)p(\theta_2 \mid \theta_{[1]}) \ldots p(\theta_m \mid \theta_{[m-1]})$ has a hierarchical structure, where $\theta_{[j]} = (\theta_1, \ldots, \theta_j)$ and $p(\theta_j \mid \theta_{[j-1]})$ denote the density of the conditional distribution of θ_j given $\theta_{[j-1]}$. Then, we perturb each level of $p(\theta)$ such that $p(\theta \mid \omega) = p(\theta_1 \mid \omega_1)$ $p(\theta_2 \mid \theta_{[1]}, \omega_2) \ldots p(\theta_m \mid \theta_{[m-1]}, \omega_m)$, $\int p(\theta_1 \mid \omega_1)d\theta_1 = 1$ and $\int p(\theta_j \mid \theta_{[j-1]}, \omega_j)d\theta_j = 1$ for

$j=2,\ldots,m$. In this case, $T_\omega\mathcal{M}$ is spanned by the m functions $\partial_{\omega_1}\log p(\theta_1\mid\omega_1)$ and $\partial_{\omega_j}\log p(\theta_j\mid\theta_{[j-1]},\omega_j)$ for $j=2,\ldots,m$. Moreover, $G_{jk}(\omega)=0$ for all $j\neq k$. For instance, it can be shown that $G_{12}(\omega)=\int\partial_{\omega_1}\log p(\theta_1\mid\omega_1)\partial_{\omega_2}\log p(\theta_2\mid\theta_{[1]},\omega_2)p(\theta\mid\omega)d\theta=\partial_{\omega_1}\partial_{\omega_2}\int p(\theta_1\mid\omega_1)p(\theta_2\mid\theta_1,\omega_2)d\theta_2 d\theta_1=\partial_{\omega_1}\partial_{\omega_2}1=0$. Thus, different components of ω are orthogonal to each other (Zhu et al., 2007). Furthermore, it follows from (4) that $G_{11}(\omega)=\int\{\partial_{\omega_1}\log p(\theta_1\mid\omega_1)\}^2 p(\theta_1\mid\omega)d\theta_1$ and $G_{jj}(\omega)=\int\{\partial_{\omega_j}\log p(\theta_j\mid\theta_{[j-1]},\omega_j)\}^2 p(\theta_j\mid\theta_{[j-1]},\omega)d\theta_j$ for $j\geqslant 2$.

Combining the above results, we are led to the following proposition, whose proof can be found in the Supplementary Material.

PROPOSITION 1. *Consider any Bayesian perturbation model to the prior given by $\mathcal{M}=\{p(\theta\mid\omega)p(z\mid\theta):\omega\in\Omega\}$. If ω is independent of z, then the metric tensor of its Bayesian perturbation manifold $\mathcal{M}$ is independent of the specification of the sampling distribution $p(z\mid\theta)$.*

Proposition 1 has important implications. The independence property ensures that existing results on local robustness to the prior can be considered as a special case of the new method developed here (McCulloch, 1989; Gustafson, 1996).

Example 4. Consider a Bayesian perturbation model given by

$$\mathcal{M}=\left\{p(z,\theta\mid\omega)=p(\theta\mid\omega_p)p(z\mid\theta,\omega_s):\omega=(\omega_p^{\mathrm{T}},\omega_s^{\mathrm{T}})^{\mathrm{T}},\int p(\theta\mid\omega_p)d\theta\right.$$
$$\left.=\int p(z;\theta,\omega_s)dz=1\right\},$$

in which $\omega_p=(\omega_1,\ldots,\omega_m)^{\mathrm{T}}$ and $\omega_s=(\omega_{m+1},\ldots,\omega_{m+n})^{\mathrm{T}}$ are assumed to be independent of both z and θ. We consider $\omega(t)=(\omega_1,\ldots,\omega_{j-1},\omega_j+t,\omega_{j+1},\ldots,\omega_{m+n})^{\mathrm{T}}$ with $\omega(0)=\omega$ for each $j\in\{1,\ldots,m+n\}$. Thus, $\dot\omega_k(0)=d\omega_k(0)/dt=1$ for $k=j$ and 0 otherwise. Letting $\ell(\theta\mid\omega_p)=\log p(\theta\mid\omega_p)$ and $\ell(z\mid\theta,\omega_s)=\log p(z\mid\theta,\omega_s)$, we have

$$\dot\ell\{z,\theta\mid\omega(0)\}=\sum_{k=1}^{m+n}\dot\omega_k(0)\partial_{\omega_k}\log p(z,\theta\mid\omega)=\partial_{\omega_j}\ell(\theta\mid\omega_p)+\partial_{\omega_j}\ell(z\mid\theta,\omega_s).\tag{5}$$

Since ω_s and ω_p have no components in common, $T_\omega\mathcal{M}$ is spanned by $m+n$ functions including $\partial_{\omega_j}\ell(\theta\mid\omega_p)$ for $j=1,\ldots,m$ and $\partial_{\omega_j}\ell(z\mid\theta,\omega_s)$ for $j=m+1,\ldots,m+n$. Note that $\int\partial_{\omega_k}\ell(\theta\mid\omega_p)\partial_{\omega_j}\ell(z\mid\theta,\omega_s)p(z,\theta\mid\omega)dzd\theta=\int\partial_{\omega_k}p(\theta\mid\omega_p)\partial_{\omega_j}p(z\mid\theta,\omega_s)dzd\theta=\partial_{\omega_k}1\partial_{\omega_j}1=0$ holds for any j,k. Therefore, it follows from (5) that the inner product of $\partial_{\omega_j}\ell(z,\theta\mid\omega)$ and $\partial_{\omega_k}\ell(z,\theta\mid\omega)$, denoted by $G_{jk}(\omega)$, is

$$\int\partial_{\omega_j}\ell(\theta\mid\omega_p)\partial_{\omega_k}\ell(\theta\mid\omega_p)p(z,\theta\mid\omega)dzd\theta+\int\partial_{\omega_j}\ell(z\mid\theta,\omega_s)\partial_{\omega_k}\ell(z\mid\theta,\omega_s)p(z,\theta\mid\omega)dzd\theta.\tag{6}$$

Moreover, the first term of (6) can be simplified to $\int\partial_{\omega_j}\ell(\theta\mid\omega_p)\partial_{\omega_k}\ell(\theta\mid\omega_p)p(\theta\mid\omega_p)d\theta$ since $\int p(z\mid\theta,\omega_s)dz=1$. For $j=1,\ldots,m$ and $k=m+1,\ldots,m+n$, it follows from (6) that $<\partial_{\omega_j}\ell(z,\theta\mid\omega),\partial_{\omega_k}\ell(z,\theta\mid\omega)>=0$ since $\partial_{\omega_k}\ell(\theta\mid\omega_p)=0$ and $\partial_{\omega_j}\ell(z\mid\theta,\omega_s)=0$. Thus, ω_s and ω_p are orthogonal to each other with respect to $<\partial_{\omega_j}\ell(z,\theta\mid\omega),\partial_{\omega_k}\ell(z,\theta\mid\omega)>$.

Combining the above results, we obtain the following proposition.

Proposition 2. *Consider* $\mathcal{M}=\{p(z,\theta\mid\omega)=p(\theta\mid\omega_p)p(z\mid\theta,\omega_s):\omega=(\omega_p^{\mathrm{T}},\omega_s^{\mathrm{T}})^{\mathrm{T}}\}$. *Assume that* ω_p *is independent of* z *and* $\int p(\theta\mid\omega_p)d\theta=\int p(z\mid\theta,\omega_s)dz=1$. *Consider two smooth curves* $p\{z,\theta\mid\omega_{(k)}(t)\}$ *with* $\omega_{(k)}(t)=\{\omega_{(k),p}(t),\omega_{(k),s}(t)\}^{\mathrm{T}}$ *such that* $\omega_{(1)}(0)=\omega_{(2)}(0)=\omega$ *and* $\omega_{(1),p}(t)$ *and* $\omega_{(2),s}(t)$ *are independent of* t. *For any two tangent vectors* $v_k(\omega)=\dot{\ell}\{z,\theta\mid\omega_{(k)}(0)\}\in T_\omega\mathcal{M}$ *for* $k=1,2$, *we have* $<v_1,v_2>(\omega)=0$.

Proposition 2 has important implications. For simultaneous perturbations to the prior and the sampling distribution, it ensures that ω_p and ω_s are geometrically orthogonal to each other. Thus, we can separate out the influence of the prior from that of the data and the sampling distribution.

Finally, we consider a simultaneous perturbation model, denoted by $p(z,\theta\mid\omega_p,\omega_d,\omega_s)$, in which ω_p, ω_d and ω_s represent individual perturbations to the prior, the data and the sampling distribution, respectively. In addition to Propositions 1 and 2, we can obtain the following theorem.

Theorem 1. *Let* $\mathcal{M}=\{p(z,\theta\mid\omega)=p(\theta\mid\omega_p)p(z\mid\theta,\omega_d,\omega_s):\omega=(\omega_p,\omega_d,\omega_s)\}$ *with* $\int p(\theta\mid\omega_p)d\theta=\int p(z\mid\theta,\omega_d,\omega_s)dz=1$ *and that* ω_p *is independent of* z. *Consider two smooth curves* $p\{z,\theta\mid\omega_{(k)}(t)\}$ *with* $\omega_{(k)}(t)=\{\omega_{(k),p}(t),\omega_{(k),d}(t),\omega_{(k),s}(t)\}^{\mathrm{T}}$ *passing through* $\omega_{(1)}(0)=\omega_{(2)}(0)=\omega$ *and having two tangent vectors* $v_k(\omega)=\dot{\ell}\{z,\theta\mid\omega_{(k)}(0)\}\in T_\omega\mathcal{M}$, $k=1,2$. *Then*:

(i) *if* $\omega_{(1),p}(t)$ *and* $\{\omega_{(2),d}(t),\omega_{(2),s}(t)\}$ *are independent of* t, *then* $<v_1,v_2>(\omega)=0$;
(ii) *if* $\{\omega_{(1),p}(t),\omega_{(1),d}(t)\}$ *and* $\{\omega_{(2),p}(t),\omega_{(2),s}(t)\}$ *are independent of* t *and* $p(z\mid\theta,\omega_d,\omega_s)=p_1(z\mid\theta,\omega_d)p_2(z\mid\theta,\omega_s)$ *for any* (ω_d,ω_s), *then* $<v_1,v_2>(\omega)=0$.

For simultaneous perturbations to the prior, the data, and the sampling distribution, Theorem 1 (i) ensures that ω_p and (ω_d,ω_s) are geometrically orthogonal to each other. If $p(z\mid\theta,\omega_d,\omega_s)=p_1(z\mid\theta,\omega_d)p_2(z\mid\theta,\omega_s)$, then ω_p, ω_d, and ω_s are geometrically orthogonal to each other.

3. Influence measures and their properties

3·1. *Intrinsic influence measures*

We consider some objective functions, such as the ϕ-divergence function, the posterior mean, and the Bayes factor, and develop associated intrinsic influence measures for quantifying the effects of perturbing the three key elements of a Bayesian analysis. An objective function of interest for sensitivity analysis is often chosen to be a functional of the perturbed posterior distribution of θ given z, given by $p(\theta\mid z,\omega)=p(z,\theta\mid\omega)/\int p(z,\theta\mid\omega)d\theta$ and $p(\theta\mid z,\omega^0)$, which is the unperturbed posterior distribution of θ given z. Such an objective function, denoted by $f(\omega,\omega^0)=f\{p(\theta\mid z,\omega),p(\theta\mid z,\omega^0)\}$, can be also regarded as a mapping from $\mathcal{M}\times\mathcal{M}$ to R. Throughout the paper, we assume that $f(\omega,\omega^0)$ is a smooth function of ω and is a path-independent function of $p(\theta\mid z,\omega)$ and $p(\theta\mid z,\omega^0)$ such that $f(\omega,\omega)=0$ for any $\omega\in\Omega$. For instance, $f(\omega,\omega^0)$ can be set as the total variation distance of $p(\theta\mid z,\omega^0)$ and $p(\theta\mid z,\omega)$ (Dey et al., 1996). Most standard influence measures such as the range (Berger, 1990, 1994) can be regarded as special cases of $f(\omega,\omega^0)$.

A large value of these influence measures can be caused by both the perturbation ω to the baseline distribution regardless of the observed data and the discrepancies between the observed data and the fitted model $p(z,\theta)$. Since the purpose of any influence analysis is to detect the discrepancies between the observed data and $p(z,\theta)$, we suggest rescaling $f(\omega,\omega^0)$ by using the shortest distance between $p(z,\theta\mid\omega)$ and $p(z,\theta\mid\omega^0)$. We explicitly quantify the distance between $p(z,\theta\mid\omega)$ and $p(z,\theta\mid\omega^0)$ by using their minimal geodesic distance, denoted by $d(\omega,\omega^0)$. If

$\mathcal{M}$ is a complete and finite-dimensional Riemannian manifold, then the Hopf–Rinow theorem states that any two points on $\mathcal{M}$ can be joined by a minimal geodesic (Ekeland, 1978). Furthermore, if $\mathcal{M}$ is a complete infinite-dimensional Riemannian manifold, any two points on $\mathcal{M}$ can be joined by a path which is almost a minimal geodesic (Ekeland, 1978). We introduce an intrinsic influence measure for comparing ω and $\omega^0 \in \Omega$ as follows. Geometrically, an intrinsic measure is invariant to certain reparameterizations.

DEFINITION 2. *The intrinsic influence measure for comparing* $p(\theta \mid z, \omega)$ *to* $p(\theta \mid z, \omega^0)$ *is defined as* $\mathrm{IGI}_f(\omega, \omega^0) = f(\omega, \omega^0)^2/d(\omega, \omega^0)^2$.

The proposed $\mathrm{IGI}_f(\omega, \omega^0)$ can be interpreted as the ratio of the change of the objective function relative to the minimal distance between $p(z, \theta \mid \omega)$ and $p(z, \theta \mid \omega^0)$ on $\mathcal{M}$. Since $f(\omega, \omega^0)$ is path-independent and $d(\omega, \omega^0)$ is invariant to smooth reparametrization of ω, $\mathrm{IGI}_f(\omega, \omega^0)$ is also invariant. Moreover, we suggest identifying the most influential ω in Ω, denoted by $\hat{\omega}_I$, which maximizes $\mathrm{IGI}_f(\omega, \omega^0)$ for all $\omega \in \Omega$.

Example 5. We consider the logarithm $\mathrm{BF}(\omega, \omega^0) = \log \int p(z, \theta \mid \omega) d\theta - \log \int p(z, \theta \mid \omega^0) d\theta$ of the Bayes factor for comparing $p(z \mid \theta, \omega)$ and $p(z \mid \theta, \omega^0)$, which can be regarded as a statistic for testing hypotheses of ω against ω^0 (Kass & Raftery, 1995). Under mild conditions, $\mathrm{BF}(\omega, \omega^0)$ is a smooth mapping from $\mathcal{M}$ to R. We can set $f(\omega, \omega^0) = \mathrm{BF}(\omega, \omega^0)$ and obtain the intrinsic influence measure

$$\mathrm{IGI}_{BF}(\omega, \omega^0) = \frac{\mathrm{BF}(\omega, \omega^0)^2}{d(\omega, \omega^0)^2}.$$

3·2. *First-order local influence measures*

We consider the local behaviour of $f\{\omega(t), \omega^0\}$ as t approaches 0 along all possible smooth curves $p\{z, \theta \mid \omega(t)\}$ passing through ω^0, that is $\omega(0) = \omega^0$. Since $f\{\omega(t), \omega^0\}$ is a function from R to R, it follows by Taylor's series expansion that $f\{\omega(t), \omega^0\} = f\{\omega(0), \omega^0\} + \dot{f}\{\omega(0)\}t + 0{\cdot}5\ddot{f}\{\omega(0)\}t^2 + o(t^2)$, where $\dot{f}\{\omega(0)\}$ and $\ddot{f}\{\omega(0)\}$ denote the first- and second order derivatives of $f\{\omega(t), \omega^0\}$ with respect to t evaluated at $t = 0$. We need to distinguish between $\dot{f}\{\omega(0)\} \neq 0$ for some smooth curves $\omega(t)$ and $\dot{f}\{\omega(0)\} = 0$ for all smooth curves $\omega(t)$. We first consider the case $\dot{f}\{\omega(0)\} \neq 0$ for some smooth curves $\omega(t)$. Let $\dot{\ell}\{z, \theta \mid \omega(0)\} = v \in T_{\omega(0)}\mathcal{M}$. Then, $\dot{f}\{\omega(0)\} = df[v]\{\omega(0)\}$ is the directional derivative of f in the direction of $v \in T_{\omega(0)}\mathcal{M}$ (Lang, 1995). We are led to the following definition.

DEFINITION 3. *The first-order local influence measure is defined as* $\mathrm{FI}_f[v]\{\omega(0)\} = \lim_{t\to 0} \mathrm{IGI}_f\{\omega(0), \omega(t)\} = [df[v]\{\omega(0)\}]^2/[< v, v > \{\omega(0)\}]$.

To carry out a sensitivity analysis, we use the tangent vector $v_{F,\max}$ in $T_{\omega(0)}\mathcal{M}$, which maximizes $\mathrm{FI}_f[v]\{\omega(0)\}$ and is invariant to reparameterization of ω(t). We now have the following result.

THEOREM 2. *The quantity* $\mathrm{FI}_f[v]\{\omega(0)\}$ *is invariant to smooth reparameterization of* $\omega(t)$.

In addition to the invariance property in Theorem 2, $\mathrm{FI}_f[v]\{\omega(0)\}$ is a direct generalization of the first-order measure for a finite-dimensional perturbation manifold (Zhu et al., 2007; Wu & Luo, 1993).

Example 5 (*continued*). We set $f\{\omega(t), \omega^0\} = \mathrm{BF}\{\omega(t), \omega^0\}$. Since $d[\mathrm{BF}\{\omega(t), \omega^0\}]/dt = \int \dot{\ell}\{z, \theta \mid \omega(0)\}[p\{z, \theta \mid \omega(0)\}/\int p\{z, \theta \mid \omega(0)\}d\theta]d\theta = \int \dot{\ell}\{z, \theta \mid \omega(0)\}p\{\theta \mid z, \omega(0)\}d\theta$, we have

$$\mathrm{FI}_f[v]\{\omega(0)\} = \frac{[\int \dot{\ell}\{z,\theta \mid \omega(0)\}p\{\theta \mid z, \omega(0)\}d\theta]^2}{\int \dot{\ell}\{z,\theta \mid \omega(0)\}^2 p\{z,\theta \mid \omega(0)\}dz d\theta}.$$

It is relatively easy to compute $\mathrm{FI}_f[v]\{\omega(0)\}$ for a specific perturbation. For instance, for the contamination to the prior given by $p\{\theta \mid \omega(t)\} = p(\theta) + t\{g(\theta) - p(\theta)\}$, it can be shown that

$$\mathrm{FI}_f[v]\{\omega(0)\} = \frac{(\int [g(\theta)\{p(\theta)\}^{-1} - 1]p\{\theta \mid z, \omega(0)\}d\theta)^2}{\int [g(\theta)\{p(\theta)\}^{-1} - 1]^2 p(\theta)d\theta} = \frac{[p_g(z)\{p(z)\}^{-1} - 1]^2}{\int [g(\theta)\{p(\theta)\}^{-1} - 1]^2 p(\theta)d\theta},$$

where $p(z) = \int p(z \mid \theta)p(\theta)d\theta$ and $p_g(z) = \int g(\theta)p\{z \mid \theta, \omega(0)\}d\theta$. Since the ratio of $p_g(z)$ to $p(z)$ is the Bayes factor in favour of $g(\theta)$ versus $p(\theta)$, $\mathrm{FI}_f[v]\{\omega(0)\}$ is the square of the normalized Bayes factor of $g(\theta)$ versus $p(\theta)$.

Example 6. Consider the Bayesian perturbation manifold $\mathcal{M} = \{p(z,\theta \mid \omega) : \omega \in \Omega \subset R^m\}$ and $p\{z,\theta \mid \omega(t)\}$ as a smooth curve on $\mathcal{M}$, in which ω is not a function of z and θ, such as the perturbation scheme in the mean-shift model, and $\omega(t) = \{\omega_1(t), \ldots, \omega_m(t)\}^{\mathrm{T}}$ is a smooth vector of t. Let $v_h = (v_{h,1}, \ldots, v_{h,m})^{\mathrm{T}} = d\omega(0)/dt$. By using the chain rule, we have

$$v\{\omega(0)\} = d\ell\{z,\theta \mid \omega(t)\}/dt \mid_{t=0} = \sum_{k=1}^{m} \dot{\omega}_k(t)\partial_{\omega_k}\ell\{z,\theta \mid \omega(0)\} = \sum_{k=1}^{m} v_{h,k}\partial_{\omega_k}\ell\{z,\theta \mid \omega(0)\},$$

$$df[v]\{\omega(0)\} = df\{\omega(t), \omega^0\}/dt \mid_{t=0} = \sum_{k=1}^{m} v_{h,k}\partial_{\omega_k} f\{\omega(0)\} = v_h^{\mathrm{T}}\partial_\omega f\{\omega(0)\}, \tag{7}$$

$$\begin{aligned} < v, v > \{\omega(0)\} &= \sum_{j,k=1}^{m} v_{h,j}v_{h,k} < \partial_{\omega_j}\ell\{z,\theta \mid \omega(0)\}, \partial_{\omega_k}\ell\{z,\theta \mid \omega(0)\} > \{\omega(0)\} \\ &= v_h^{\mathrm{T}} G\{\omega(0)\}v_h, \end{aligned}$$

where $\partial_{\omega_k} f(\omega)$ denotes the first-order partial derivative of $f(\omega, \omega^0)$ with respect to ω_k and $G\{\omega(0)\} = \int [\partial_\omega \ell\{z,\theta \mid \omega(0)\}]^{\otimes 2} p(z,\theta \mid \omega)dz d\theta$ is an $m \times m$ Fisher information matrix with respect to ω. Thus, it follows from (7) and the definition of $\mathrm{FI}_f[v]\{\omega(0)\}$ that $\mathrm{FI}_f[v]\{\omega(0)\} = [df[v]\{\omega(0)\}]^2/[< v, v > \{\omega(0)\}] = [v_h^{\mathrm{T}}\partial_\omega f\{\omega(0)\}]^2/v_h^{\mathrm{T}} G\{\omega(0)\}v_h$. Finally, we obtain $v_{F,\max}\{\omega(0)\} = \mathrm{argmax}_v \mathrm{FI}_f[v]\{\omega(0)\} = [G\{\omega(0)\}]^{-1/2}\partial_\omega f\{\omega(0)\}$.

3·3. *Second-order local influence measures*

We use $\ddot{f}\{\omega(0)\}$ to assess the second-order local influence of ω to a statistical model (Zhu et al., 2007). However, for a general smooth curve $\omega(t)$ on $\mathcal{M}$, $\ddot{f}\{\omega(0)\}$ is not geometrically well behaved (Lang, 1995; Zhu et al., 2007). We consider only the geodesic $p\{z,\theta \mid \omega(t)\}$, denoted by $\mathrm{Exp}_{\omega(0)}(tv)$, passing through $\mathrm{Exp}_{\omega(0)}(tv) \mid_{t=0} = \omega(0)$ with initial direction $\dot{\ell}\{z,\theta \mid \omega(0)\} = v\{\omega(0)\} \in T_{\omega(0)}\mathcal{M}$. It follows from a Taylor's series expansion (Lang, 1995; Zhu et al., 2007) that

$$f\{\mathrm{Exp}_{\omega(0)}(tv), \omega^0\} = f\{\omega(0), \omega^0\} + t\,df[v]\{\omega(0)\} + 0{\cdot}5t^2\ddot{f}\{\mathrm{Exp}_{\omega(0)}(tv)\} \mid_{t=0} + o(t^2), \tag{8}$$

where $\ddot{f}\{\mathrm{Exp}_{\omega(0)}(tv)\} = d^2 f\{\mathrm{Exp}_{\omega(0)}(tv), \omega^0\}/dt^2$. Geometrically, $\ddot{f}\{\mathrm{Exp}_{\omega(0)}(tv)\} \mid_{t=0}$ in (8) is called the Riemannian Hessian and is denoted by $\mathrm{Hess}(f)(v,v)\{\omega(0)\}$ (Lang, 1995). The Riemannian Hessian is symmetric. We now introduce a second-order influence measure.

DEFINITION 4. *The second-order influence measure in the direction $v \in T_{\omega(0)}\mathcal{M}$ is defined as* $\mathrm{SI}_f[v]\{\omega(0)\} = \mathrm{Hess}(f)(v, v)\{\omega(0)\}/[< v, v > \{\omega(0)\}]$.

Geometrically, $\mathrm{SI}_f[v]\{\omega(0)\}$ is invariant to scalar transformations and smooth transformations. To carry out a sensitivity analysis, we use the tangent vector $v_{S,\max} \in T_{\omega(0)}\mathcal{M}$, which maximizes $\mathrm{SI}_f[v]\{\omega(0)\}$ for all $v \in T_{\omega(0)}\mathcal{M}$. There is a direct connection between the second-order measures in finite- and infinite-dimensional spaces. Therefore, the diagnostic method proposed here can be regarded as an extension of existing local influence approaches (Cook, 1986; Zhu et al., 2007) to an infinite dimensional setting.

Example 6, *continued*. We consider the Bayesian perturbation model in Example 6. If $df[v]\{\omega(0)\} = 0$ for all $v \in T_{\omega(0)}\mathcal{M}$, then $\mathrm{Hess}(f)(v, v)\{\omega(0)\}$ reduces to $v_h^{\mathrm{T}} H_f\{\omega(0)\} v_h$, where $H_f\{\omega(0)\} = \partial_\omega^2 f\{\omega(0)\}$, in which $\partial_\omega^2 f\{\omega(0)\}$ denotes the second-order partial derivative of $f(\omega, \omega^0)$ with respect to ω (Zhu et al., 2007). In this case, $\mathrm{SI}_f[v]\{\omega(0)\} = v_h^{\mathrm{T}} H_f(\omega, \omega^0) v_h / v_h^{\mathrm{T}} G\{\omega(0)\} v_h$ and $v_{S,\max}$ equals the eigenvector of $G(\omega)^{-1/2} H_f\{\omega(0)\} G(\omega)^{-1/2}$ corresponding to its largest eigenvalue. Let e_j be an $m \times 1$ vector with jth element 1 and 0 otherwise. We also suggest an index plot of $\mathrm{SI}_f[e_j]$ to examine influential cases (Zhu et al., 2007, p. 2572).

3·4. *Bayesian influence analysis*

We now summarize the four key steps in carrying out our proposed influence analysis.

Step 1. Construct a Bayesian perturbation model $p(z, \theta \mid \omega)$.

Step 2. Given the Bayesian perturbation model, we calculate the geometric quantities, such as $< v, v > \{\omega(0)\}$, of the perturbation manifold.

Step 3. Choose an objective function $f(\omega, \omega^0)$ and calculate $\mathrm{IGI}_f(\omega, \omega^0)$ and $\hat{\omega}_I = \mathrm{argmax}_{\omega\in\Omega} \mathrm{IGI}_f(\omega, \omega^0)$.

In Step 3, we need to compute $f(\omega, \omega^0)$ and $d(\omega, \omega^0)$. Since $f(\omega, \omega^0)$ is a function of $p(\theta \mid z, \omega)$ and $p(\theta \mid z, \omega^0)$, we use Markov chain Monte Carlo methods to draw random samples from $p(\theta \mid z, \omega)$ and $p(\theta \mid z, \omega^0)$ and then evaluate $f(\omega, \omega^0)$ (Chen et al., 2000). We use the Dijkstra algorithm (Dijkstra, 1959) to approximate the geodesic distance between $p(z, \theta \mid \omega)$ and $p(z, \theta \mid \omega^0)$. The main idea of this method is to discretize the model $\{p(z, \theta \mid \omega) : \omega \in \Omega\}$ into a simpler space $\{p(z, \theta \mid \omega) : \omega \in \Omega_D\}$, where Ω_D contains a set of the refined grid points of Ω and then we approximate $d(\omega, \omega^0)$ (Dijkstra, 1959). Based on the set of the refined grid points Ω_D, we then calculate $\{\mathrm{IGI}_f(\omega, \omega^0) : \omega \in \Omega_D\}$ and approximate $\hat{\omega}_I$ by using $\mathrm{argmax}_{\omega\in\Omega_D}$ $\mathrm{IGI}_f(\omega, \omega^0)$.

Step 4. If $df[v]\{\omega(0)\} \neq 0$, then we calculate $v_{F,\max}$ to assess local influence of minor perturbations to the model. However, if $df[v]\{\omega(0)\}$ is 0 for all v, then we compute $\mathrm{SI}_f[v]\{\omega(0)\}$ and find $v_{S,\max} = \mathrm{argmax}[\mathrm{SI}_f[v]\{\omega(0)\}]$.

In Step 4, we need to compute $\mathrm{FI}_f[v]\{\omega(0)\}$ and $\mathrm{SI}_f[v]\{\omega(0)\}$. For many infinite-dimensional manifolds, such as the additive ϵ-contamination class, v varies in a set $\mathcal{V}$, which may be well approximated by a finite number of grid points $\{v_l : l = 1, \ldots, K_0\}$. We can approximate $\mathrm{argmax}_v[\mathrm{FI}_f[v]\{\omega(0)\}]$ and $\mathrm{argmax}_v[\mathrm{SI}_f[v]\{\omega(0)\}]$ by $\mathrm{argmax}_{v_l}[\mathrm{FI}_f[v_l]\{\omega(0)\}]$ and $\mathrm{argmax}_{v_l}[\mathrm{SI}_f[v_l]\{\omega(0)\}]$, respectively.

4. A theoretical example

We consider a dataset $z = (z_1, \ldots, z_n)^{\mathrm{T}}$ to illustrate the potential applications of our proposed diagnostics. Assume that $z_1, \ldots, z_n$ are independent and identically distributed from a $N(\theta, 1)$ distribution and the baseline prior distribution of θ is the density corresponding to a $N(\mu_0, \sigma_0^2)$ distribution. Letting $\bar{z} = \sum_{i=1}^n z_i/n$, we have $p(\theta \mid z) \propto \exp[-0{\cdot}5(n + 1/\sigma_0^2)\{\theta - (n\bar{z} + \mu_0/\sigma_0^2)/(n + 1/\sigma_0^2)\}^2]$.

We first consider a simple perturbation to the location of the baseline prior, whose perturbed model is given by

$$p(z, \theta \mid \omega) = p(z \mid \theta)p(\theta \mid \omega) = p(z \mid \theta) \exp\{-0{\cdot}5(\theta - \omega - \mu_0)^2/\sigma_0^2\}/(2\pi\sigma_0^2)^{0{\cdot}5}$$

for $\omega \in [\omega_L, \omega_U]$, where ω_L and ω_U are known scalars. We set $E(\theta \mid z, \omega) = \int \theta p(\theta \mid z, \omega) d\theta = \{n\bar{z} + (\omega + \mu_0)/\sigma_0^2\}/(n + 1/\sigma_0^2)$ and $f(\omega, \omega^0) = E(\theta \mid z, \omega) - E(\theta \mid z, \omega^0)$. Thus, following Berger (1990), we have that the range of $f(\omega, \omega^0)$ equals $f(\omega_U, \omega^0) - f(\omega_L, \omega^0) = (\omega_U - \omega_L)/(n\sigma_0^2 + 1)$. A large range can be caused by a large $\omega_U - \omega_L$, which is associated with the size of the perturbation to the prior, as shown later.

We compute the intrinsic structure of $p(z, \theta \mid \omega)$ and the intrinsic influence measure. We can calculate the geodesic distance between $p(z, \theta \mid \omega_L)$ and $p(z, \theta \mid \omega_U)$. Since $\omega(t) = t$ and $\dot{\ell}\{z, \theta \mid \omega(t)\} = (\theta - \mu_0 - t)/\sigma_0^2$, we have $< \dot{\ell}\{z, \theta \mid \omega(t)\}, \dot{\ell}\{z, \theta \mid \omega(t)\} > \{\omega(t)\} = 1/\sigma_0^2$ and $d(\omega_L, \omega_U) = \int_{\omega_L}^{\omega_U} dt/\sigma_0 = (\omega_U - \omega_L)/\sigma_0$, which is the size of the sole perturbation to the prior regardless of the data. Both small σ_0 and large $\omega_U - \omega_L$ can introduce large perturbations. When $f(\omega, \omega^0) = E(\theta \mid z, \omega) - E(\theta \mid z, \omega^0)$, we have $\mathrm{IGI}_f(\omega, \omega^0) = \sigma_0^2/(n\sigma_0^2 + 1)^2$, which is independent of ω. This indicates that relative to the perturbation of the prior, $f(\omega, \omega^0)$ does not change too much. A large range gives a false indication of the extent of nonrobustness, which is actually caused by large perturbations to the prior (Sivaganesan, 2000).

Secondly, we consider a simultaneous perturbation to the prior and the model, given by

$$p(z, \theta \mid \omega) \propto \exp\left\{-0{\cdot}5\sum_{i=1}^n (z_i - \omega_i - \theta)^2 - 0{\cdot}5(\theta - \mu_0 - \omega_{n+1})^2/\sigma_0^2\right\}, \tag{9}$$

where $\omega = (\omega_1, \ldots, \omega_{n+1})^{\mathrm{T}} \in R^{n+1}$. In this case, $\omega^0 = 0_{n+1}$ represents no perturbation. Let δ_{ij} equal 1 for $i = j$ and 0 otherwise. Following Example 6, we can show that for $i, j = 1, \ldots, n$,

$$\begin{aligned}
&\partial_{\omega_i}\ell(z, \theta \mid \omega) = (z_i - \omega_i - \theta), \quad \partial_{\omega_{n+1}}\ell(z, \theta \mid \omega) = (\theta - \mu_0 - \omega_{n+1})/\sigma_0^2,\\
&< \partial_{\omega_i}\ell(z, \theta \mid \omega), \partial_{\omega_j}\ell(z, \theta \mid \omega) > (\omega) = \delta_{ij}, \quad < \partial_{\omega_i}\ell(z, \theta \mid \omega), \partial_{\omega_{n+1}}\ell(z, \theta \mid \omega) > (\omega) = 0,\\
&< \partial_{\omega_{n+1}}\ell(z, \theta \mid \omega), \partial_{\omega_{n+1}}\ell(z, \theta \mid \omega) > (\omega) = 1/\sigma_0^2.
\end{aligned} \tag{10}$$

Thus, when $\sigma_0 \neq 1$, ω_i for $i = 1, \ldots, n$ and ω_{n+1} introduce different levels of perturbation to the fitted model $p(z, \theta \mid \omega)$. Furthermore, since $< \partial_{\omega_i}\ell(z, \theta \mid \omega), \partial_{\omega_j}\ell(z, \theta \mid \omega) > (\omega)$ for all i, j are independent of ω, the manifold $\mathcal{M}$ determined by (9) is a flat manifold (Lang, 1995). For any ω in R^{n+1}, the geodesic connecting $p(z, \theta \mid \omega)$ and $p(z, \theta \mid \omega^0)$ is given by $p(z, \theta; t\omega)$ for $t \in [0, 1]$. By using (3), we can show that $d(\omega, \omega^0)^2 = \sum_{i=1}^n \omega_i^2 + \omega_{n+1}^2/\sigma_0^2$, which quantifies the size of the perturbation scheme (9) to the prior and the fitted model.

We calculate the logarithm of the Bayes factor $\mathrm{BF}(\omega, \omega^0)$ as discussed in Example 5. Since the terms in the exponential function of (9) form a quadratic function of θ, we can explicitly calculate $\mathrm{BF}(\omega, \omega^0) = P(\omega) - P(\omega^0)$, where $P(\omega) = \log \int p(z, \theta \mid \omega) d\theta$ equals

$$C - 0{\cdot}5\left[(\omega_{n+1} + \mu_0)^2/\sigma_0^2 + \sum_{i=1}^{n}(z_i - \omega_i)^2 - \left\{(\omega_{n+1} + \mu_0)\Big/\sigma_0^2 + \sum_{i=1}^{n}(z_i - \omega_i)\right\}^2\Big/(n + 1/\sigma_0^2)\right],$$

and C is a scalar independent of ω. Now recall the results of Example 5. For a smooth curve $\omega(t) \in R^{n+1}$ with $\omega(0) = \omega^0$, $\mathrm{FI}_f[v]\{\omega(0)\}$ is determined by $\partial_\omega \mathrm{BF}(\omega, \omega^0)$ and $v_{F,\max}(\omega) = \{G(\omega^0)\}^{-1/2}\partial_\omega \mathrm{BF}(\omega, \omega^0)$, in which $G(\omega^0) = \mathrm{diag}(1, \ldots, 1, \sigma_0^{-2})$ as calculated in (10). Taking derivatives of $\mathrm{BF}(\omega, \omega^0)$ with respect to ω, we get

$$\partial_{\omega_{n+1}}\mathrm{BF}(\omega, \omega^0) = -(\omega_{n+1} + \mu_0)/\sigma_0^2 + \{(\omega_{n+1} + \mu_0)/\sigma_0^2 + \sum_{i=1}^{n}(z_i - \omega_i)\}/(n\sigma_0^2 + 1),$$

$$\partial_{\omega_i}\mathrm{BF}(\omega, \omega^0) = z_i - \omega_i - \{(\omega_{n+1} + \mu_0)/\sigma_0^2 + \sum_{i=1}^{n}(z_i - \omega_i)\}/(n + 1/\sigma_0^2)$$

for $i = 1, \ldots, n$, which yields

$$v_{F,\max}(\omega^0) = \left\{z_1 - \frac{n\bar{z} + \mu_0/\sigma_0^2}{n + 1/\sigma_0^2}, \ldots, z_n - \frac{n\bar{z} + \mu_0/\sigma_0^2}{n + 1/\sigma_0^2}, \frac{n(\bar{z} - \mu_0)\sigma_0}{n\sigma_0^2 + 1}\right\}^{\mathrm{T}}. \tag{11}$$

By inspecting the first n components of $v_{F,\max}(\omega^0)$, we can identify outlying points z_i which are far from the posterior mean of θ, while the last component of $v_{F,\max}(\omega^0)$ can pick up an influential hyperparameter μ_0.

Thirdly, we consider a simultaneous perturbation to the prior and the sampling distribution,

$$p(z, \theta \mid \omega) \propto \exp\left\{-0{\cdot}5\sum_{i=1}^{n}\omega_i(z_i - \theta)^2 - 0{\cdot}5\omega_{n+1}(\theta - \mu_0)^2/\sigma_0^2 + 0{\cdot}5\sum_{i=1}^{n+1}\log(\omega_i)\right\},$$

where $\omega = (\omega_1, \ldots, \omega_{n+1})^{\mathrm{T}} \in R^{n+1}$. In this case, $\omega^0 = 1_{n+1}$ represents no perturbation. Following Example 6, we can show that for $i, j = 1, \ldots, n$,

$$\begin{aligned}
\partial_{\omega_i}\ell(z, \theta \mid \omega) &= -0{\cdot}5(z_i - \theta)^2 + 0{\cdot}5\omega_i^{-1},\\
\partial_{\omega_{n+1}}\ell(z, \theta \mid \omega) &= -0{\cdot}5(\theta - \mu_0)^2/\sigma_0^2 + 0{\cdot}5\omega_{n+1}^{-1},\\
< \partial_{\omega_i}\ell(z, \theta \mid \omega), \partial_{\omega_j}\ell(z, \theta \mid \omega) > (\omega) &= 0{\cdot}5\omega_i^{-2}\delta_{ij},\\
< \partial_{\omega_i}\ell(z, \theta \mid \omega), \partial_{\omega_{n+1}}\ell(z, \theta \mid \omega) > (\omega) &= 0,\\
< \partial_{\omega_{n+1}}\ell(z, \theta \mid \omega), \partial_{\omega_{n+1}}\ell(z, \theta \mid \omega) > (\omega) &= 0{\cdot}5\omega_{n+1}^{-2}.
\end{aligned} \tag{12}$$

Thus, $G(\omega^0)$ is an $(n + 1) \times (n + 1)$ identity matrix.

We consider a sensitivity analysis for predictive distributions (Lavine, 1992; Millar & Stewart, 2007). Let z_{n+1} denote a future observation from $N(\theta, 1)$, the predictive density of z_{n+1} given z, denoted by $p(z_{n+1} \mid z, \omega)$, is shown to be $N\{(\sum_{i=1}^n \omega_i z_i + \omega_{n+1}\mu_0/\sigma_0^2)/(\sum_{i=1}^n \omega_i + \omega_{n+1}/\sigma_0^2), 1/(\sum_{i=1}^n \omega_i + \omega_{n+1}/\sigma_0^2)\}$. We set $f(\omega, \omega^0) = \int z_{n+1}p(z_{n+1} \mid z, \omega)dz_{n+1} - \int z_{n+1}p(z_{n+1} \mid z, \omega^0)dz_{n+1}$. Now recall the results of Example 6 and the metric tensor in (12). For a smooth curve $\omega(t) \in R^{n+1}$ with $\omega(0) = \omega^0$, $\mathrm{FI}_f[v]\{\omega(0)\}$ is determined by $\partial_\omega f(\omega)$ and

$v_{F,\max}(\omega) = \partial_\omega f(\omega, \omega^0)$, which are given by

$$\partial_{\omega_{n+1}} f(\omega, \omega^0) = \sigma_0^{-2}\mu_0 \bigg/ \left(\sum_{i=1}^n \omega_i + \omega_{n+1}/\sigma_0^2\right) - \sigma_0^{-2}\left(\omega_{n+1}\mu_0/\sigma_0^2 + \sum_{i=1}^n z_i\omega_i\right) \bigg/ \left(\sum_{i=1}^n \omega_i + \omega_{n+1}/\sigma_0^2\right)^2,$$

$$\partial_{\omega_i} f(\omega, \omega^0) = z_i \bigg/ \left(\sum_{i=1}^n \omega_i + \omega_{n+1}/\sigma_0^2\right) - \left(\omega_{n+1}\mu_0/\sigma_0^2 + \sum_{i=1}^n z_i\omega_i\right) \bigg/ \left(\sum_{i=1}^n \omega_i + \omega_{n+1}/\sigma_0^2\right)^2$$

for $i = 1, \ldots, n$. This yields that $v_{F,\max}(\omega^0)$ is proportional to

$$\frac{1}{n + 1/\sigma_0^2}\left(z_1 - \frac{n\bar{z} + \mu_0/\sigma_0^2}{n + 1/\sigma_0^2}, \ldots, z_n - \frac{n\bar{z} + \mu_0/\sigma_0^2}{n + 1/\sigma_0^2}, \frac{n(\mu_0 - \bar{z})\sigma_0^2}{n\sigma_0^2 + 1}\right)^{\mathrm{T}}. \qquad (13)$$

We observe that $v_{F,\max}(\omega^0)$ in (13) is closely associated with $v_{F,\max}(\omega^0)$ in (11), and thus $v_{F,\max}(\omega^0)$ is able to pick up outlying points z_i and an influential hyperparameter μ_0.

Finally, we examine a more general setting in which z_i $(i = 1, \ldots, 50)$ are independent $N(\theta_i, 1)$ variables, with the θ_i independently generated from a Dirichlet process prior $DP(c_0F_1)$, where the base measure F_1 is that of a $N(5, 1)$ distribution and the confidence parameter c_0 is set equal to 2 (Escobar, 1994). Furthermore, the z_i were changed to $z_i + 5$ for $i = 49$ and 50, which can be regarded as two outliers. We fit a model with $z_i \sim N(\theta_i, 1)$ and $\theta_i \sim DP(2F_0)$, where F_0 is the probability measure of a $N(0, 1)$ distribution. The base measure F_0 is misspecified due to the difference between the means of a $N(0, 1)$ and the true base measure $N(5, 1)$. We consider a simultaneous perturbation to the prior and the data. We have

$$p(z, \theta \mid \omega) \propto \exp\left(-0{\cdot}5\sum_{i=1}^n (z_i - \omega_i - \theta_i)^2 + \sum_{i=1}^n \log\left[c_0F_0(\theta_i) + c_0\omega_{n+1}\{F_1(\theta_i) - F_0(\theta_i)\} + \sum_{j=1}^{i-1}\delta_{\theta_j}(\theta_i)\right]\right). \qquad (14)$$

In this case, $\omega^0 = 0_{n+1}$ represents no perturbation. By differentiating $\ell(z, \theta \mid \omega) = \log p(z, \theta \mid \omega)$ in (14) with respect to each component of ω, we have that for $i = 1, \ldots, n$,

$$\partial_{\omega_i}\ell(z, \theta; \omega) = z_i - \omega_i - \theta_i,$$

$$\partial_{\omega_{n+1}}\ell(z, \theta \mid \omega) = \sum_{i=1}^n \frac{c_0\{F_1(\theta_i) - F_0(\theta_i)\}}{c_0F_0(\theta_i) + c_0\omega_{n+1}\{F_1(\theta_i) - F_0(\theta_i)\} + \sum_{j=1}^{i-1}\delta_{\theta_j}(\theta_i)}.$$

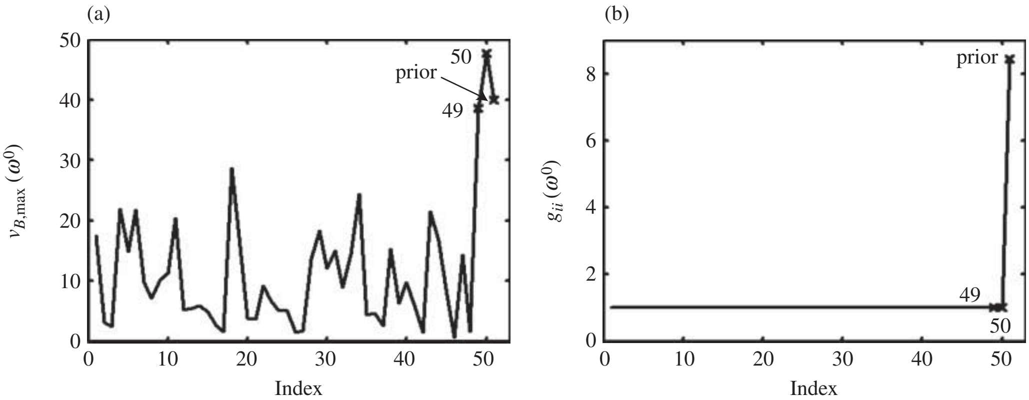

Fig. 1. Simultaneous perturbation model using a Dirichlet process prior and perturbing individual observations: (a) local influence measures $v_{B,\max}(\omega^0)$ for the logarithm of the Bayes factor $f(\omega,\omega^0)=\mathrm{BF}(\omega,\omega^0)$, from which the outlying cases 49 and 50 and the perturbation to the Dirichlet process prior were detected; (b) index plot of metric tensor $g_{ii}(\omega^0)$ for the perturbation (15).

Since $\int(z_i-\omega_i-\theta_i)p(z,\theta\mid\omega)dz=0$ and $\int(z_i-\omega_i-\theta_i)(z_j-\omega_j-\theta_j)p(z,\theta\mid\omega)dz=\delta_{ij}$, we have

$$< \partial_{\omega_i}\ell(z,\theta\mid\omega), \partial_{\omega_j}\ell(z,\theta\mid\omega) > (\omega)=\delta_{ij}, \quad < \partial_{\omega_i}\ell(z,\theta\mid\omega), \partial_{\omega_{n+1}}\ell(z,\theta\mid\omega) > (\omega)=0,$$

$$< \partial_{\omega_{n+1}}\ell(z,\theta\mid\omega), \partial_{\omega_{n+1}}\ell(z,\theta\mid\omega) > (\omega)=E[\{\partial_{\omega_{n+1}}\ell(z,\theta\mid\omega)\}^2].$$

Similar to (11), we set $f(\omega,\omega^0)=\mathrm{BF}(\omega,\omega^0)$ and substitute the results from (7) to calculate $v_{F,\max}(\omega^0)$ using 50 000 Markov chain Monte Carlo samples generated from the posterior distribution $p(\theta_1,\ldots,\theta_n\mid z_1,\ldots,z_{50})$ after a 5000 sample burn-in. Inspecting the components of $v_{F,\max}(\omega^0)$ reveals the outlying cases 49 and 50 and shows the sensitivity to the misspecified base measure F_0 of the Dirichlet process prior for θ_i in Fig. 1.

In addition to this theoretical example, an extensive simulation and a real data analysis involving missing data are given in the Supplementary Material. In practice, we suggest an iterative process to carry out the four-step influence analysis in § 3·4. If one is concerned about sensitivity to the prior, then one may introduce some finite-dimensional perturbation as in Example 1 to all hyperparameters of the prior and identify influential hyperparameters according to their local influence measures. Then, for a few influential hyperparameters, one further perturbs their associated prior distribution using the additive ϵ-contamination class and then carries out intrinsic influence analysis. If one is concerned about the sampling distribution, then one may introduce various perturbations including the additive ϵ-contamination class and the perturbation model (1) to $p(z\mid\theta)$ and use the local influence measures to detect which part of $p(z\mid\theta)$ is sensitive to minor perturbations. Then, one may focus on these influential parts and carry out an intrinsic influence analysis. After refining the prior and the sampling distribution, one may then perturb individual observations and detect a set of influential observations. After examining the information from each influence analysis, we carry out a simultaneous perturbation to z, $p(\theta)$ and $p(z\mid\theta)$. We start with a local influence analysis to examine the sensitivity of all components and then focus on a few influential components using an intrinsic influence analysis.

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Supplementary material

Supplementary Material available at *Biometrika* online includes the proof of Proposition 1, a real data analysis on missing data problems and an extensive simulation.

Appendix

Proof of Proposition 1. Consider any two smooth curves $p\{z,\theta \mid \omega_{(k)}(t)\} = p\{\theta \mid \omega_{(k)}(t)\}p(z \mid \theta)$ with $p\{z,\theta \mid \omega_{(k)}(0)\} = p(\theta \mid \omega)p(z \mid \theta)$ for $k = 1, 2$. For each k, by differentiating $\ell\{z,\theta \mid \omega_{(k)}(t)\}$ with respect to t, we obtain a tangent vector $v_k(\omega) = \dot{\ell}\{z,\theta \mid \omega_{(k)}(0)\} = d\log p\{\theta \mid \omega_{(k)}(t)\}/dt\mid_{t=0} \in T_\omega\mathcal{M}$, which is independent of $p(z \mid \theta)$. Furthermore, letting $d_t = d/dt$, the inner product of $v_1(\omega)$ and $v_2(\omega)$ is given by $\int [d_t \log p\{\theta \mid \omega_{(1)}(t)\}][d_t \log p\{\theta \mid \omega_{(2)}(t)\}]p\{z,\theta \mid \omega\}dz d\theta = \int [d_t \log p\{\theta \mid \omega_{(1)}(t)\}][d_t \log p\{\theta \mid \omega_{(2)}(t)\}]p\{\theta \mid \omega\}d\theta$, which is also independent of $p(z \mid \theta)$. □

Proof of Proposition 2. Consider two smooth curves $p\{z,\theta \mid \omega_{(k)}(t)\}$ with $\omega_{(k)}(t) = \{\omega_{(k),p}(t)^{\mathrm{T}}, \omega_{(k),s}(t)^{\mathrm{T}}\}^{\mathrm{T}}$ such that $\omega_{(1)}(0) = \omega_{(2)}(0) = \omega$ and $\omega_{(1),p}(t)$ and $\omega_{(2),s}(t)$ are independent of t. Let $\ell(z \mid \theta, \omega_{(1),s}) = \log p(z \mid \theta, \omega_{(1),s})$. Since $\omega_{(1),p}(t)$ is independent of t,

$$v_1(\omega) = \dot{\ell}\{z,\theta \mid \omega_{(1)}(0)\} = \frac{d}{dt}\log p\{\theta \mid \omega_{(1),p}(t)\}\mid_{t=0} + \frac{d}{dt}\log p\{z \mid \theta, \omega_{(1),s}(t)\}\mid_{t=0} = \dot{\ell}\{z \mid \theta, \omega_{(1),s}(0)\}.$$

Let $\ell(\theta \mid \omega_{(2),p}) = \log p(\theta \mid \omega_{(2),p})$. Similarly, we have

$$v_2(\omega) = \dot{\ell}\{z,\theta \mid \omega_{(2)}(0)\} = \frac{d}{dt}\log p\{\theta \mid \omega_{(2),p}(t)\}\mid_{t=0} = \dot{\ell}\{\theta \mid \omega_{(2),p}(0)\}.$$

Thus, the inner product of $v_1(\omega)$ and $v_2(\omega)$, denoted by $< v_1, v_2 > (\omega)$, is given by

$$\begin{aligned}\int \dot{\ell}\{\theta \mid \omega_{(2),p}(0)\}\dot{\ell}\{z \mid \theta, \omega_{(1),s}(0)\}p(z,\theta \mid \omega)dz d\theta &= \int \frac{dp\{\theta \mid \omega_{(2),p}(0)\}}{dt}\frac{dp\{z \mid \theta, \omega_{(1),s}(0)\}}{dt}dz d\theta \\ &= \int \left(\frac{dp\{\theta \mid \omega_{(2),p}(0)\}}{dt}\left[\int \frac{dp\{z \mid \theta, \omega_{(1),s}(0)\}}{dt}dz\right]\right)d\theta \\ &= \int \left(\frac{dp\{\theta \mid \omega_{(2),p}(0)\}}{dt}\frac{d[\int p\{z \mid \theta, \omega_{(1),s}(0)\}dz]}{dt}\right)d\theta \\ &= \int \left[\frac{dp\{\theta \mid \omega_{(2),p}(0)\}}{dt}\frac{d1}{dt}\right]d\theta = 0.\end{aligned}$$ □

Proof of Theorem 1. Since Theorem 1 (i) follows from Proposition 2, we focus on Theorem 1 (ii). Since $\{\omega_{(1),p}(t), \omega_{(1),d}(t)\}$ and $\{\omega_{(2),p}(t), \omega_{(2),s}(t)\}$ are independent of t and $p(z \mid \theta, \omega_d, \omega_s) = p_1(z \mid \theta, \omega_d)p_2(z \mid \theta, \omega_s)$, we have

$$v_1(\omega) = \dot{\ell}\{z,\theta \mid \omega_{(1)}(0)\} = \frac{d}{dt}\log p_1\{z \mid \theta, \omega_{(1),s}(t)\}\mid_{t=0},$$

$$v_2(\omega) = \dot{\ell}\{z,\theta \mid \omega_{(2)}(0)\} = \frac{d}{dt'}\log p_2\{z \mid \theta, \omega_{(2),d}(t')\}\mid_{t'=0}.$$

Thus, $< v_1, v_2 > (\omega)$ is given by

$$\begin{aligned}&\int \left.\frac{d\log p_1\{z \mid \theta, \omega_{(1),s}(t)\}}{dt}\right|_{t=0}\left.\frac{d\log p_2\{z \mid \theta, \omega_{(2),d}(t')\}}{dt'}\right|_{t'=0}p(z,\theta \mid \omega)dz d\theta \\ &\quad= \int \frac{dp_1\{z \mid \theta, \omega_{(1),s}(0)\}}{dt}\frac{dp_2\{z \mid \theta, \omega_{(2),d}(0)\}}{dt'}p(\theta \mid \omega_p)dz d\theta = \frac{d^2 1}{dt dt'} = 0.\end{aligned}$$ □

Proof of Theorem 2. Consider a smooth curve $p\{z,\theta \mid \omega(t)\}$. Let $R(s) : [c_1, c_2] \to [-\epsilon, \epsilon]$ be the first-order differential map such that $R(c_3) = 0$ and $\dot{R}(c_3) = dR(s)/ds\mid_{s=c_3} \neq 0$ for a $c_3 \in (c_1, c_2)$. Then,

$p[z, \theta \mid \omega\{R(s)\}]$ is a differential map from $[c_1, c_2]$ to $\mathcal{M}$. It follows from the chain rule that $\dot{f}[\omega\{R(s)\}] = d_s f[\omega\{R(s)\}, \omega^0] = d_r f\{\omega(r), \omega^0\}\dot{R}(s)$ and $d_s \ell[z, \theta \mid \omega\{R(s)\}] = d_r \ell\{z, \theta \mid \omega(r)\}\dot{R}(s)$, where $\dot{R}(s) = d_s R(s)$, $d_c = d/dc$, $d_r = d/dr$, and $d_s = d/ds$. Thus, as $\omega(0) = \omega^0$, we have

$$df[\dot{R}(c_3)v][\omega\{R(c_3)\}] = \dot{R}(c_3) df[v](\omega), \text{ and } < \dot{R}(c_3)v, \dot{R}(c_3)v > (\omega) = \dot{R}(c_3)^2 < v, v > (\omega). \quad \square$$

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